Differentiation

Def: $f : U \mapsto \mathbb{R}$, U open, $U \subset \mathbb{R}$, $x_0 \in U$

f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

This limit is called the derivative of f at x_0 and it is denoted by

$$f'(x_0) = \frac{df}{dx}(x_0).$$

Example:

(1) $f(x) = 3x$

$$\frac{f(x) - f(x_0)}{x - x_0} = 3 \Rightarrow f'(x_0) = 3$$

(2) $f(x) = x^2$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} (x + x_0)$$

$$= 2x_0 . = f'(x_0)$$

Obs: $f'(x)$ exists $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$.

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \varepsilon |x - x_0|$.

i.e. $|f(x) - L(x)| < \varepsilon |x - x_0|$, $\forall |x - x_0| < \delta$.

Let $L(x) = f(x_0) + f'(x_0)(x - x_0)$ is a linear function.

$f'(x_0)$ exists $\Leftrightarrow$ $L(x)$ is a "good" approx. of $f(x)$ near x_0.

Proposition $U \subset \mathbb{R}$, U open, $f: U \mapsto \mathbb{R}$, $x_0 \in U$. f differentiable at x_0 $\Longrightarrow$ f continuous at x_0.

~~proof~~: Since f is differentiable at x_0.
Let $\alpha > 0$, $\exists\, \beta > 0$ s.t. $|x - x_0| < \beta$
$\Rightarrow |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \alpha |x - x_0|$

$$
\begin{aligned}
|f(x) - f(x_0)| &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| \\
&\quad + |f'(x_0)(x - x_0)| \\
&\leq \alpha |x - x_0| + |f'(x_0)|\,|x - x_0| \\
&= (\alpha + |f'(x_0)|)\,|x - x_0|
\end{aligned}
$$

Let $\varepsilon > 0$, select $\delta = \min\left\{ \beta, \frac{\varepsilon}{\alpha + |f'(x_0)|} \right\}$.
then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

<u>Def</u>: $U \in \mathbb{R}$, U open, $f : U \mapsto \mathbb{R}$.
f is differentiable if f is differentiable at x, $\forall x \in U$.

Example: $f(x) = |x|$ is continuous at $x=0$ but not differentiable at $x=0$.

Check, $\lim_{x\to 0^-} f(x) = -1$ while $\lim_{x\to 0^+} f(x) = +1$.

Proposition $f, g : U \mapsto R$, $U \subset \mathbb{R}$ U open. f and g are differentiable at $x_0 \in U$, $c \in \mathbb{R}$. then,

(1) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

(2) $(cf)'(x_0) = c \cdot f'(x_0)$

(3) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

(4) $\left(\frac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$

if $g(x_0) \neq 0$

proof: (3) $(fg)(x) - (fg)(x_0) = f(x)g(x) - f(x_0)g(x_0) = (f(x)g(x) - f(x_0)g(x)) + (f(x_0)g(x) - f(x_0)g(x_0)) = g(x)(f(x) - f(x_0)) + f(x_0)(g(x) - g(x_0))$.

So, $\dfrac{(fg)(x)-(fg)(x_0)}{x-x_0} = g(x)\cdot \dfrac{f(x)-f(x_0)}{x-x_0}$

$+ f(x_0) \dfrac{g(x)-g(x_0)}{x-x_0}$.

take the limit $x - x_0$

$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$

<u>Corollary</u> $n \in \mathbb{Z}^+$, $\dfrac{d}{dx} x^n = n x^{n-1}$.

~~proof~~ : $n=1$, $\dfrac{d}{dx} x' = 1$ ✓

$n \geqslant 1$, $x^{n+1} = x^n \cdot x$

$\dfrac{d}{dx} x^{n+1} = \dfrac{d}{dx} x^n \cdot x + x^n \cdot \dfrac{d}{dx} x$

$= n x^{n-1} \cdot x + x^n$

$= (n+1) x^n$.